Advanced Math: Notes on Lessons 94-97

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Lesson 94: Graphs of Secant and Cosecant / Graphs of Tangent and Cotangent

To graph secant (reciprocal of cosine) and cosecant (reciprocal of sine), first lightly draw (say with a dashed line) its reciprocal (the cosine or sine), and then draw the final one. They will have distinctive "U" shaped graphs, and to positive/negative infinity where their reciprocals go to zero. See the book for the tangent/cotangent graphs.

Lesson 95: Advanced Complex Roots

This is essentially a repeat of lesson 79, where we learned how to find the roots of complex numbers.

Basically, if you have a complex number and want to find its roots, first write it in polar form (R cis A), and then reverse De Moivre's theorem as:

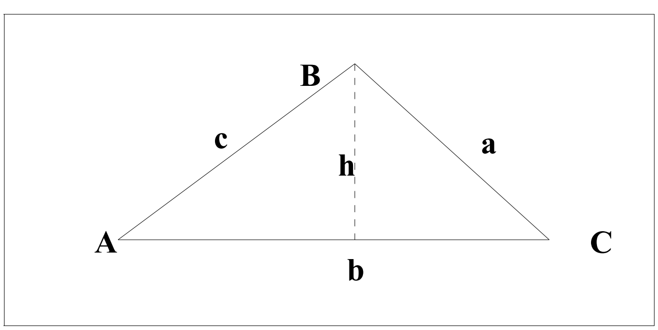
$$\sqrt[n]{a}=\sqrt[n]{r\,cis\,\theta}=\sqrt[n]{r}\,cis(\theta/n) \quad because \quad a^n=r^n\,cis(n\theta)$$

The other solutions all have the same magnitude; just keep adding 360°/n. (The angles of all of the nth roots differ by 360°/n). Remember that for any nth root, you will have n solutions.

Be careful when calculating the angle of the original complex value; it's easy to get that wrong if either the real or imaginary parts are negative. Many calculators have an "atan2" function or a "rectangular to polar" function. If you have it, use it when you can; it will determine the angle for you and avoid a common source of error (most calculators can't give exact answers though). E.G., on TI-83 and TI-84 calculators, the function R►Pθ(x,y), acquired using 2nd ANGLE 6, will compute the angle given the x (real) and y (imaginary) parts (to compute the range, use R►Pr). Be sure that you have the intended angle measure (degrees, radians, or gradians). OpenOffice.org calls the function "atan2(x;y)" - again it gives you the angle. You then divide it by n to find the angle of the n^{th} root, and keep adding/subtracting 360°/n to get all n answers.

Lesson 96: More double-angle identities / triangle area formula / proof of the law of sines / equal angles imply proportional sides

Remember that the area of a triangle is ½(base)(height). With trigonometry, given two lengths of two sides and the angle opposite the third, we can easily find a general equation for the area which uses the sine of the angle.



sin A = h/c, so h = c sin A

area = ½(base)(height) = ½(b)(c sin A)

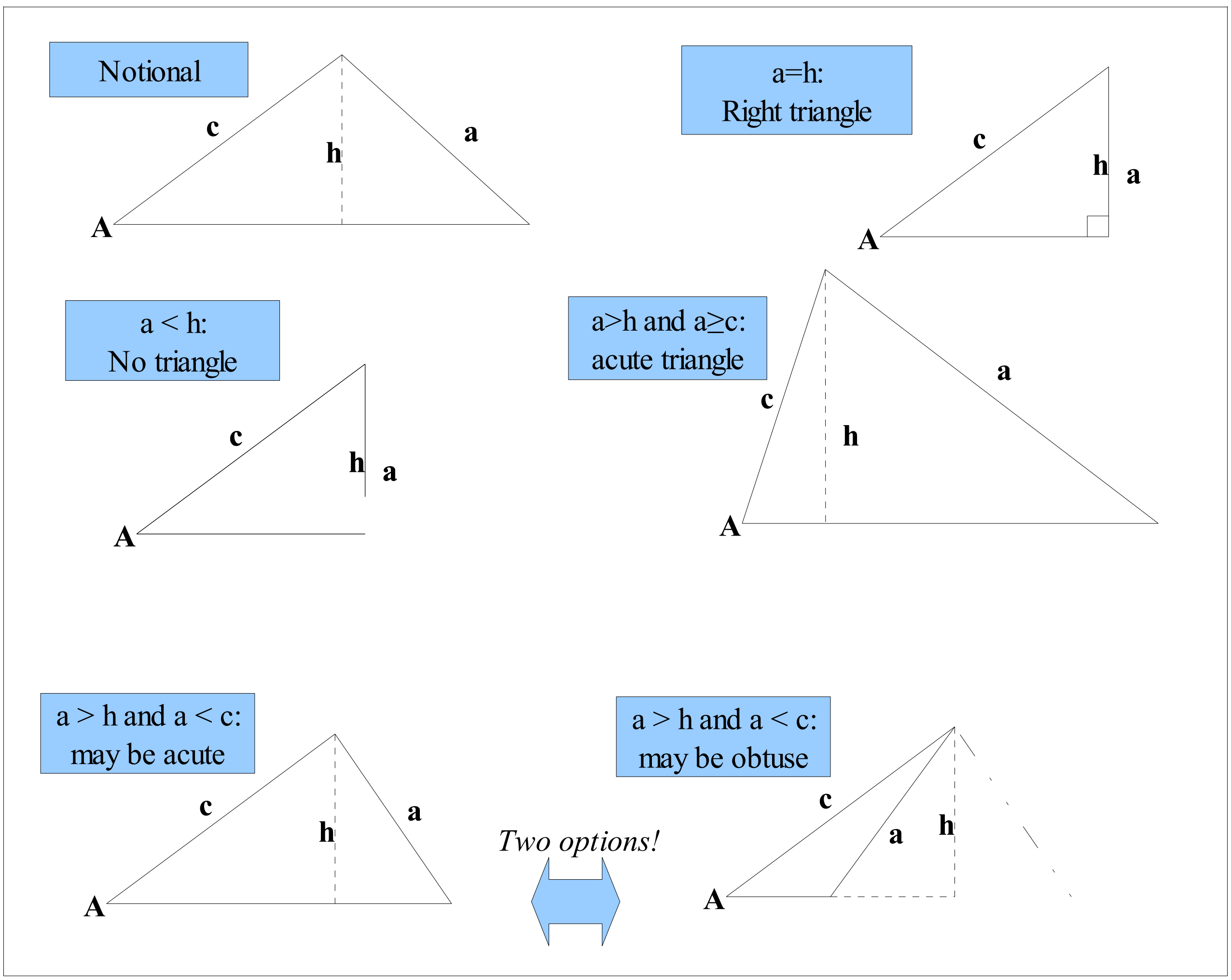
area = ½ bc sin A

If two different triangles have equal angles, then they are proportional - that is, the sides of one are equal to the sides of the other each multiplied by the the same constant P.

Lesson 97: The Ambiguous Case (for triangles)

Let's say you're told you have a triangle, and you have an angle (call it A), the length of its opposite side (call it a), and another side's length (call it c)... but no figure. This is trickier than it might sound, because without a figure, you don't know what kind of triangle you have. In some cases, two different triangles could meet that specification!

The solution is to find the height of the triangle, and then compare its value to the values of the sides a and c. Using the previous figure, height h = c sin A. There are then 4 possible outcomes; I've ordered the last three differently from Saxon, in the hopes that this will be easier to understand. The problem is that if a>h and a<c, we don't know if we have an acute triangle or an obtuse triangle (both are possible). Also: Saxon forgets one case: a>h and a=c (this is acute).



Worked Examples for Trig Identities

Individual trig identities aren't complicated, but using them can be. Part of the problem is that each identity has limited power. More importantly, though, there's no "algorithm" that automatically solves arbitrary trig expressions. Instead, you have recognize various patterns that might suggest certain ways will solve a problem (or will solve it more easily). I know of no one who has a good way of teaching this other than "do lots of problems and practice" - as you do so, you'll start to recognize those patterns.

So with that in mind, we'll solve several problems from the book, and I'll discuss tactics for why one approach or the other might be effective. Unless you're specifically told to use some technique (which is rare), it doesn't matter how you solve a problem, as long as each step is legitimate. We still can't divide by zero, but we will temporarily ignore that for a little while, so that you can gain experience in doing the problems (eventually we'll have to re-instate that rule, and when it asks for specific numerical answers, you still need to check for legality).

Lesson 96, Problem Set #5

$$\frac{\cos^4 x - \sin^4 x}{\cos 2\mathrm{x}} \stackrel{?}{=} 1$$

Note the question mark above the equal sign; this clearly indicates that the equality of the left and right hand sides is unknown (and to be proved), and *not* something we can take for granted. In the denominator we have "cos 2x"; there aren't many identities that can use "cos 2x", so it's usually better to convert that using the cos 2x = $\cos^2$ x – $\sin^2$ x identity (since we have lots of identities involving cos x and sin x). In the numerator you should recognize that we have something matching the form $A^2 - B^2$. This expands the left-hand-side to:

$$\frac{(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)}{\cos^2 x - \sin^2 x}$$

The denominator and the right-hand-side of the numerator cancel out, and $\cos^2$ x + $\sin^2$ x equals 1. Thus we end up with 1.

Lesson 96, Problem Set #6

$$(\sin x + \cos x)^2 \stackrel{?}{=} 1 + \sin 2\mathrm{x}$$

Expand out both sides, and solve. Once again we'll use $\cos^2$ x + $\sin^2$ x equals 1.

$$\sin^2 x + 2\sin x \cos x + \cos^2 x \stackrel{?}{=} 1 + 2\sin x \cos x$$
$$1 + 2\sin x \cos x = 1 + 2\sin x \cos x \qquad \text{QED}$$

The "QED" at the end is an abbreviation of the Latin phrase "quod erat demonstrandum", which literally means, "that which was to be demonstrated". The phrase is a translation into Latin of the original Greek ὅπερ ἔδει δειξαι (hoper edei deixai) used by Euclid, Archimedes, and other early mathematicians. QED often appears at the end of a mathematical proof or philosophical argument to show that the proof is complete. Modern texts often indicate the end of a proof with ■, which is alternatively called "solid black square", "tombstone", or "Halmos symbol" (after Paul Halmos).

Lesson 96, Problem Set #7

$$\frac{1-\cos x}{\sin x} \stackrel{?}{=} \frac{\sin x}{1+\cos x}$$

If we had "1 – $\cos^2$ x" on the left-hand-side numerator, we could quickly simplify it to "$\sin^2$ x". But we don't, and it's not obvious how we could convert to it. So let's just move both denominators to the other side, and see if we can solve it (we can):

$$(1+\cos x)(1-\cos x) \stackrel{?}{=} (\sin x)(\sin x)$$

$$1-\cos^2 x \stackrel{?}{=} \sin^2 x \qquad \text{QED}$$

Lesson 97, Problem Set #8

$$\cos 2\text{x} + 2\sin^2 x \stackrel{?}{=} 1$$

Again, there's little we can do with "cos 2x" other than expand it, so we'll do that. Once we do, it simplifies immediately to the answer: $\cos 2\text{x} + 2\sin^2 x = (\cos^2 x - \sin^2 x) + 2\sin^x = \cos^2 x + \sin^2 x = 1$

Lesson 97, Problem Set #9

$$\frac{\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x} \stackrel{?}{=} 2\tan x$$

We could expand the "tan x" term, but it's not obvious that this would help – all the complication is on the left-hand-side! So let's try to simplify the left-hand-side. We have two fractions; to add or subtract them, we have to have equal denominators. So we'll try to simplify the left-hand-side, starting with making the denominators equal so that we can subtract the numerators:

$$\frac{\cos x(1+\sin x)}{(1-\sin x)(1+\sin x)} - \frac{\cos x(1-\sin x)}{(1+\sin x)(1-\sin x)}$$

$$\frac{\cos x(1+\sin x) - \cos x(1-\sin x)}{(1+\sin x)(1-\sin x)}$$

$$\frac{\cos x + \cos x \sin x - \cos x + \cos x \sin x}{1-\sin^2 x}$$

$$\frac{2\cos x \sin x}{\cos^2 x}$$

$$\frac{2\sin x}{\cos x}$$

$$2\tan x$$

And so we've proved it! QED.