Advanced Math: Notes on Lessons 82-85

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Lesson 82: Taking the Logarithm of / Exponential Equations

You "take the logarithm of something" by applying a log function with some base. For example, if you have 1000, you can show taking the logarithm of 1000 as log 1000. Note that $\log_{10} 1000 = 3$, because the result of $\log_{10} x$ = whatever you raise 10 by to get x.

If you have an equation where the *exponent* has the variable, you can take the logarithm to extract it, by using the fact that $\log_b x^n = n \log_b x$. For example:

$$\begin{aligned} 100^{2x+1} &= 1000000 \\ \log_{10} 100^{2x+1} &= \log_{10} 1000000 \\ (2x+1)\log_{10} 100 &= 6 \\ (2x+1)2 &= 6 \\ x &= 1 \end{aligned}$$

You should use whatever logarithm base makes the problem easy to solve; e.g., choose b=10 if x is 10 or some exponent of 10, choose b=e when x is e, and so on. Choosing b=x is often easiest, but many calculators don't have a simple way to take logs of arbitrary bases, and you haven't been shown yet how to do them. (We'll get there.)

Note: Saxon (and many other books) presume that if you don't specify the log base, it's 10. That's a common, but not universal, assumption.

Lesson 83: Simple Probability / Independent Events / Replacement

Part A: Simple Probability

"Probability" is simply the likelihood that some event will come true (given what we know of the situation),. from 0% to 100% likelihood. The simplest kinds of probability come from coin flips, dice rolls, and the like, so we'll use them often as examples. Unless otherwise stated, from here on we'll assume that these are fair with exactly equal probabilities of their possible outcomes, and that dice are 6-sided. When all possible outcomes have the same likelihood, we can calculate probability this way:

$$\textit{Probability of event } E = P(E) = \frac{\textit{number of outcomes that are } E}{\textit{total number of possible outcomes}}$$

So if we have a (fair) coin and flip it, there are two outcomes: Heads or Tails. The probability of showing tails in a coin flip is:

$$P(tails) = \frac{1}{2} = 50\%$$

In a (6-sided) die, the probability of rolling a 5 or greater is:

$$P(\text{5 or greater}) = \frac{2}{6} = \frac{1}{3} = 33\frac{1}{3}\%$$

For problems that involve the sum of two six-sided dice, you usually need to consult a table that adds them up. Here's such a table:

Sum	***1***	***2***	***3***	***4***	***5***	***6***
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Notice that each of the unshaded cells shows a sum, and each one of those cells has equal probability.

So the probability that you'll roll the sum of 4 or less is equal to:

$$P(4 \text{ or less}) = \frac{Event(4) + Event(3) + Event(2)}{36} = \frac{(3+2+1)}{36} = \frac{6}{36} = \frac{1}{6}$$

Independent Events

If the events do not affect each other, they are called independent events. For example, if we both roll a die, the outcome of my roll should not affect yours.

The probability of independent events occurring in a specific order is the product of their individual probabilities. E.G., the probability that I will roll a "3", and you will roll a "6", on a six-sided die, is:

P(first) · P(second) = (1/6) (1/6) = 1/36

Replacement

If you repeatedly take cards from a deck, balls from an urn, etc., and do not return (replace) them, then the events are *not* independent. For example, if you take an Ace from a deck, and don't return it, that clearly reduces the chance of getting an Ace the next time!

The probability of non-independent events occurring in a specific order is still the product of their individual probabilities, **but** you will need to recalculate the probabilities for each event (because the probability will change).

E.G., given an urn with 3 red balls, and 3 green balls, the probability that you will retrieve two red balls depends on whether or not you return (place) the ball after the first draw:

If you replace them, the events are independent. (3/6)(3/6) = 1/4 (25%).

If you do not replace them, the events are not independent: (3/6)(2/5) = 6/30 = 1/5 (20%). Note that on the second draw, we had one fewer ball that met the criteria, and we had one fewer ball in total.

Lesson 84: Factorable Trig Expressions / Sketching Sinusoids

Factorable Trig Expressions

To solve trig expressions you often need to factor them. Trig expressions factor just like any other expression, but because they're longer, you might not recognize them at first. So, here you'll get practice in recognizing them.

Simple Expression		***Trig Expression***	
Original	***Factored***	***Original***	***Factored***
x^3	xx^2	$\sin^3 x$	$\sin x \sin^2 x$
$x^2 - 1$	$(x+1)(x-1)$	$\tan^2 x - 1$	$(\tan x + 1)(\tan x - 1)$
		$\csc^2 x - 1$	$(\csc x + 1)(\csc x - 1)$
$1 - x^2$	$(1 + x)(1 - x)$	$1 - \tan^2 x$	$(1 + \tan x)(1 - \tan x)$
$x^2 - y^2$	$(x + y)(x - y)$	$\csc^4 x - \cot^4 x$	$(\csc^2 x + \cot^2 x) (\csc^2 x - \cot^2 x)$
$x - xy$	$x(1 - y)$	$\sin x - \sin x \cos^2 x$	$\sin x (1 - \cos^2 x)$
$1 - 2x + x^2$	$(1 - x)(1 - x)$	$1 - 2\sin^2 x + \sin^4 x$	$(1 - \sin^2 x)(1 - \sin^2 x)$
$(x^3 - y^3)$	$(x - y)(x^2 + xy + y^2)$	$\sin^3 x - \cos^3 x$	$(\sin x - \cos x)(\sin^2 x + \sin x \cos x + \cos^2 x)$

In a sense, this lesson doesn't teach you anything new; you already know how to factor. But many people have trouble using the "same old rules" to trig functions; by practicing you'll get over that.

In real life, you may have to try several different identities or factorings before you can get an answer from an expression with trig functions. Expect a few false starts.

Sketching sinusoids

It's easier to draw sinusoids if you just draw the curve first, and then add the labels. You can determine the labels by figuring out the centerline, how far up/down it goes, the phase (how far it's been moved left/right), and how rapidly it repeats. See the text; I don't know how to explain it better.

Lesson 85: Advanced Trig Equations / Clock Problems

Advanced Trig Equations

To solve equations with trig functions, use factorings and identities to simplify them down. You will often have multiple solutions, so be careful to get them all... and be sure that they are all reasonable solutions. You may have to use several different techniques together to solve them!

Here's an example, let's try to solve this (a problem I've devised), for $0° \leq x < 360°$:

$$4\sin^2 x + 2 = 5\cos x$$

We have both a sine and cosine on the value we're trying to solve. Sometimes we can get closer to a solution if we use all the same trig function. We can use the fact that $\sin^2 x + \cos^2 x = 1$ to do this, since from that equation $\sin^2 x = 1 - \cos^2 x$:

$$4(1 - \cos^2 x) + 2 = 5 \cos x$$

Simplify:

$$4 - 4\cos^2 x + 2 - 5 \cos x = 0$$

$$-4\cos^2 x - 5 \cos x + 6 = 0$$

$$4\cos^2 x + 5 \cos x - 6 = 0$$

We can now use the quadratic equation; a=4, b=5, c=-6, to solve for cos x:

$$(\cos x) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{25+96}}{8} = \frac{-5 \pm 11}{8} = \left\{\frac{3}{4}, -2\right\}$$

Since cos x can only vary between -1 and 1, we can throw away the -2 as impossible.

Solving for:

$$\cos x = 3/4$$

Using the arccos key on a calculator, we can easily get the first quadrant solution:

$$x = 41.41° \text{ and } ?$$

We know that arccos only gives one solution on a calculator, but we know that cos goes through a cycle - it will hit any value twice through its cycle. How do we get the other answer? Well, since we know that cos x = cos -x, since 41.41° is a solution, then -41.41° is a solution. That's not between 0° and 360°, but you can always add 360° without changing an angle, so -41.41°+360° = 318.59°.

We can then check this:

x = 41.41°?

$4\sin^2 (41.41°) + 2 = 3.75$; $5\cos 41.41° = 3.75$. Yes!

x = 318.59°?

$4\sin^2 (318.59°) + 2 = 3.75$; $5\cos 318.59° = 3.75$. Yes!

So the solutions are x = {41.41°,318.59°}

Clock problems

Sometimes we want to measure distance in some arbitrary unit, like "spaces". E.G., on a clock, you could say each tick mark (that represents a minute) is one space away from the next one. Then you can use all your usual tools for measuring rate, time, and distance.