Advanced Math: Notes on Lessons 74-77

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Lesson 74: Cramer's Rule

Sets of linear equations are common (both because they happen often, and because more complicated equations are sometimes "too hard" and so we approximate using linear equations). There's a way to quickly solve them, called "Cramer's Rule". The rule is actually more general; we'll just examine the 2x2 (2 equations, 2 unknowns) case for now.

Given equations (x and y unknowns, rest are constants):

ax + by = e

cx + dy = f

We could use elimination to solve these: multiply the top one by "d" and the bottom one by (-b), solve for x. Or multiply the top by -c and the bottom by a, and you could solve for y. If you multiply them out, you'll find that there's a pattern called "Cramer's Rule", which uses determinants to solve this (see lesson 69). Here is Cramer's rule for the 2x2 case:

$$x=\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}=\frac{ed-fb}{ad-cb} \qquad y=\frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}=\frac{af-ce}{ad-cb}$$

Notice that the denominators are the same, it's just the list of coefficients of the unknowns. On top, it's the same list of coefficients of the unknowns, but replace the column of the variable you're solving with the right-hand constants. So for x, use "e f" instead of "a c"; for y, use "e f" instead of "b d".

As usual, we have a division - and you can't divide by zero. So if the denominator calculates to 0, there is no unique solution (either there's no solution, or there are an infinitely large number of solutions).

Lesson 75: Combinations

First, remember the term "permutation". A permutation P(n,r) takes a set of n items, and reports the number of different *ordered* sets of r items that can be taken them. Ordering is important in a permutation; "123" is different than "321".

A combination is like a permutation, but where ordering is irrelevant. In other words, a combination C(n, r) takes a set of n items, and reports the number of different sets of r items (where order doesn't matter) that can be taken from them.

There are lots of ways to notate combinations, including these:

$${}_nC_r=C_r^n=C_{n,r}=C(n,r)=\binom{n}{r}$$

One of the most popular notations is the right-hand-side one (with stacked parens), and I think it's a terrible notation - it looks way too much like a matrix. I recommend the C(n,r) notation, myself, but any of these are acceptable (here and elsewhere).

Calculating a combination is easy; just compute the permutation P(n, r), and then divide it by the number of different orderings of the resulting set r (which would be r!). This division is what gets rid of the ordering assumption built into permutations. Thus, to calculate a combination:

$$C(n,r)=\frac{P(n,r)}{r!}=\frac{n!}{(n-r)!r!}$$

So given 10 people, how many different groups of 4 can we create? Well, notice that here the order doesn't matter, so it's a combination - not a permutation. So we use the equation for combinations:

$$C(10,4)=\frac{10!}{(10-4)!4!}=\frac{10\cdot 9\cdot 8\cdot ...}{(6\cdot 5\cdot ...)(4\cdot 3\cdot 2\cdot 1)}=\frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1}=\frac{5040}{24}=210$$

Here's something you have to think through: Whenever you take an unordered set of r items from n items, you leave behind a set of unordered n-r items. This means that C(n, r) = C(n, n-r). You can prove this with algebra:

$$C(n,r)=\frac{n!}{(n-r)!r!} \quad so \quad C(n,n-r)=\frac{n!}{(n-(n-r))!(n-r)!}=\frac{n!}{r!(n-r)!}=\frac{n!}{(n-r)r!}$$

The fact that C(n, r)=C(n, n-r) can be important, because if the number n is large and the number r is small, it can be hard to calculate... but with this swap it often becomes easier. You want a "big number" as r when you calculate, because that will cancel out most of the numerator. In fact, factorials get large so fast that many spreadsheets/calculators *have* to use this trick or they can't do the calculation.

Most spreadsheets and scientific calculators have a function to compute combinations (and they'll build in this trick). For example, on OpenOffice.org, it's COMBIN(n;r).

Lesson 76: Functions of (-x), Functions of the other angle, Trig Identities, Rules of the Game

This lesson is an odd grab-bag of different things.

Part A

Part A looks complicated, but it's really simple. This just defines two types of symmetric functions:

- If f(x)=f(-x) for all x (in the domain of f), f is an *even function.* Even functions are symmetric about the y-axis (where x=0), so as you go further from x=0 in either direction you get the same value. Cosine is an even function.

- If f(-x)=-f(x) for all x (in the domain of f), f is an *odd function*. Odd functions "flip over" at (0,0), that is, they are "symmetric at the origin". Sine is an odd function.

Most functions aren't even or odd. There's a function that is both: f(x) = 0.

Part B

Part B goes through a simple argument to explain where the term "cosine" comes from. Remember that:

- the sum of a triangle's interior angles is always 180°
- In a right triangle one of those angles is always 90°
- What's "left over" is thus always 90° in a right triangle - so if one of the other angles is θ, then the third angle must be 90°-θ
- An angle that is 90°-θ of the other is a *complementary* angle

So in a right triangle, the two non-right-angles have to be complements of each other.

sin θ = opposite/hypotenuse; let's pick one of the non-right-angle angles as θ, label its opposite a, and the hypotenuse h. cos x = *its* adjacent/hypotenuse, but that other angle's "adjacent" is the same as the "opposite" of the first angle, and x= 90°-θ (because they're complementary).

Thus cos (90°-θ) = its adjacent/hypotenuse = sine's opposite/hypotenuse. In other words...

cos (90°-θ) = sin θ, for any θ.

This is where the "co" comes from; cosine is the "sine of the complementary angle". Same for cotangent and cosecant.

Part C/D: Trig identities/ Rules of the Game

The equation:

$$x + 1 = 3$$

Is obviously *not* true for any x, but only for x=2. An equation that's only true under some conditions is called a "conditional equation". In this example, the condition is that x=2; it's *not* true for x=0.

There's a different kind of equality that is *unconditional*, that is, *something that is true no matter what the variables' values are*. An equation that expresses that two expressions are equal, no matter what the variables' values are, is called an "identity". Identities are really important; if you have an identity, that means you can substitute one for another. In higher levels of math this is the fundamental way to solve problems - you repeatedly use identities to solve the problem.

Unfortunately, Saxon uses an older notation where the symbol "=" can indicate an "conditional equation" *or* "identity" - you can't tell from the symbols which one is meant. That can be really confusing. For example, to express that a number subtracted from itself is *always* zero, Saxon would write:

$$x - x = 0$$

But notice that this particular statement is true for *any* real or complex number x, not just for a particular number. We're not being asked to solve for x; we're expressing that whenever we see "x -

x” for any x, we can immediately replace it with 0.

Many people today would use a different symbol, ≡, if they mean “identity”. This symbol (the identity operator) is easy to remember - it’s just an equality with an extra horizontal line (mnenomic: “these are *really* equal”). By using different symbols for different meanings, we avoid confusion. I suggest doing the same. (It won’t be *wrong* to use Saxon’s notation, which is still relatively common, but it may confuse you.) That means that to express that a number subtracted from itself is *always* zero, I suggest writing this:

$$x - x \equiv 0$$

Where “≡” can be read as “is always equal to” or “is identical to” or “is equivalent to”.

You’ve actually been using identities for a long time. When you add two “x” values, and replace that with “2x”, you’re just exploiting the identity:

$$x + x \equiv 2x$$

Solving equations with trig functions often involves using one or more identities involving trig functions. We just examined one, so let’s restate it using the new identity symbol:

$$\cos(90°-\theta) \equiv \sin\theta$$

We also know some other identities, which define some functions:

$$\cot\theta \equiv \frac{1}{\tan\theta} \qquad sec\,\theta \equiv \frac{1}{\cos\theta} \qquad csc\,\theta \equiv \frac{1}{\sin\theta} \qquad \tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

Basically, to solve most trig problems, you have to use various trig identities to replace what you have with something else until you get to the answer. Fundamentally this isn’t different than replacing (2+3) with (5); you’re replacing some expression with something else that it’s always equal to. The trick is to figure out which replacement to use. It’s often a good idea to replace everything else with just sines and cosines - then you can often simplify things further. That’s just a useful strategy - it doesn’t work every time.

E.G., to show that cot θ sin θ is cos θ:

cot θ sin θ ; Start (Given)

(1/tan θ)(sin θ) ; Substitute definition of cot

(cos θ/sin θ)(sin θ) ; Substitute definition of tan

cos θ ; Simplify

Note: This looks fine, but there’s a problem: We used division, and you *still* can’t divide by zero. An equation only holds when you don’t divide by zero... and since our derivation process involved dividing by sin θ, our result is only proven if $\sin\theta \neq 0$. If you look at the original equation, you’ll notice that you can’t calculate cot θ when $\sin\theta = 0$, either - for the same reason. Handling trig identities is complicated enough, so the book will often gloss over avoiding division-by-zero (for now) so that you can learn the basics. Also, in some cases you can use more advanced techniques to “work around” a potential division by zero, but you don’t know how to do that yet. So for the moment, just be aware of this.

Lesson 77: Binomial Expansions

There's a pattern here:

$(x + y)^0$	1
$(x + y)^1$	$1x^1 + 1y^1$
$(x + y)^2$	$1x^2 + 2xy + 1y^2$
$(x + y)^3$	$1x^3 + 3x^2y + 3xy^2 + 1y$
$(x + y)^4$	$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$
$(x + y)^5$	$1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$

Notice the powers first, that's an easy pattern. The first exponent starts with the left-hand-side component at that power, and then decreases by one in each term. Thus, for $(x+y)^5$, the first term is x^5, the next includes x^4, and so on. The reverse happens for the left-hand side.

But what about the coefficients? That turns out to be easily calculated using "Pascal's triangle". Basically, start with "1" at the top, then on the next line put places on the left and right branches of the previous terms and add up the two left and right branches above. Like this:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

This gives you a quick way to raise two-term polynomials to a higher power.