Advanced Math: Notes on Lessons 66-69

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Lesson 66: Formulas for Systems of Equations / Phase Shifts and Period Changes

You already know how to solve simultaneous equations for a system of 2 equations with 2 unknowns. If you solve for abstract coefficients, you'll get a general solution for such a system. So given:

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

To solve for x, multiply the upper equation by b_2, the lower one by $-b_1$, and add them up; the y coefficients will cancel. To solve for y, multiply the upper equation by $-a_2$ and the bottom by a_1. E.G.:

$$(a_1 x + b_1 y = c_1)(b_2) \quad \rightarrow \quad a_1 b_2 x + b_1 b_2 y = b_2 c_1$$
$$(a_2 x + b_2 y = c_2)(-b_1) \quad \rightarrow \quad -a_2 b_1 x - b_1 b_2 y = -b_1 c_2$$

When you add these up, the "y" terms cancel, and you can solve $x = (b_2c_1 - b_1c_2)/(a_1b_2 - a_2b_1)$.

A more general form for many uses of sine and cosine are:

$$y = A + B\sin(C(\theta - D))$$
$$y = A + B\cos(C(\theta - D))$$

where:

A: Value of the centerline. The centerline itself is "y=A"

B: Amplitude (maximum distance from the centerline)

C: 360°/period or 2π/period

D: Phase shift

The "period" is the distance between repeats; this is an angle measure, so it will be measured in degrees, radians, or gradians. In "sin x", the period is 360° (2π radians). Since C=360°/period, then period=360°/C (for any value of C). Thus, given C, you can solve for the period.

Given some existing equation, you can factor it into this form to find these values. E.G., given:

y = 6 sin (3x+90°) - 4

you can factor it into:

y = -4 + 6 sin 3(x - -30°)

So A=-4, B=6, C=3, and D=-30°. Note that D is negative when the original equation adds.

Since period=360°/C and C=3, the period for this example is 360°/C=360°/3=120°.

You should also be able to write down the general form, given a figure. Be careful about the sign of the phase shift! A positive phase shift increases the value that will be *subtracted* from the input angle. You might find it easier to draw if you draw the normal sine/cosine function, and instead of starting it from 0, start it at the x value of "D".

Lesson 67: Antilogarithms

Given some base b, the **antilogarithm** of the number c is $\mathbf{b^c}$. Let's see why. Given:

$$y = \log_b x$$

then if we exponentiate both sides by base b, we get:

$$b^y = b^{\log_b x}$$
$$b^y = x$$

In short, exponentiation is the reverse of the logarithm function. The result of an exponentiation, used this way, is the *antilogarithm*. Again: given some base b, the **antilogarithm** of a number c is $\mathbf{b^c}$.

NOTE: The antilogarithm is the result of a function that requires *two* input values: the base, and the number to find the antilogarithm of. This isn't the first such function you've seen; "add" (+) is actually a function that takes two values (traditionally called "left" and "right") and produces their sum (left+right). It's okay to have a function with multiple inputs, as long as it produces no more than one output for any set of inputs.

Lesson 68: Locus definition of a parabola / Translated parabolas / Applications / Derivation

A *line* is the locus ("set") of all points in a plane that are equidistant from two designated points. Similarly, a *circle* is the locus of all points in a plane that are equidistant from a designated point (the "center"). But what's a parabola?

A *parabola* is the locus of all points that are equidistant from a given point, called the *focus*, and a given line, called the *directrix*. The focus is "inside" the parabola; the directrix is on the other side of the parabola (from the focus) and "parallel" to it.

Parabolas which have their vertex at the origin (0,0) and open up or down have this form:

$y = \frac{1}{4p} x^2$ with focus at (0,p) and directrix y=-p.

So for example, given the parabola $y = \frac{1}{8} x^2$ you can solve for p:

$$\frac{1}{8} = \frac{1}{4p}$$
$$8 = 4p$$
$$2 = p$$

Note the "reciprocate both sides" in step 2, which makes this easier.

In this example, the focus is (0,2), and the directrix is y=-2.

More generally, if we translate a parabola by h units horizontally and k units vertically (but still pointing up or down, and not at an angle), the equation of a parabola is:

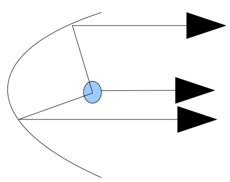
$$y - k = a\,(x - h)^2$$

A parabola whose vertex is (h,k) and focus (h,k+p) has as its equation:

$$y-k=\frac{1}{4p}(x-h)^2$$ with axis of symmetry x=h

Parabolas are widely used, for example, with sound or electromagnetic waves (including light). A parabolic reflector (a reflector in the shape of a parabola) will take an emission from the focus' position and send it out in a straight line "out" from the parabola - useful for flashlights, searchlights, and microwave antenna transmitters. Similarly, it can receive waves and focus (concentrate) them all onto the "focus" - very useful for telescopes and microwave antenna receivers.

Here's a 2D representation of a typical flashlight. The flashlight's reflector is in the shape of a parabola, with the bulb at the focus. By itself, the bulb shines line in all directions, but the reflector (because it's in the shape of a parabola) redirects all the light it catches to go in the same direction – making the flashlight's light much stronger in that direction:



Lesson 69: Matrices / Determinants

A *matrix* is a rectangular array of numbers or expressions that stand for numbers. Notationally, they may be surrounded by square brackets. The individual entries are called *elements* of the matrix. Every row must have the same number of elements. The *dimensions* (size) is listed by the number of rows, followed by the number of columns. Two matrices are equal if each element, in the same position, has the same value. Here are two 2x3 matrices, which happen to be equal:

$$\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 8-1 \end{bmatrix}$$

A matrix that has the same number of rows and columns is a *square matrix*. Every square matrix has a value called a "determinant" (matrices that aren't square don't have determinants).

Given this 2x2 square matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant is ad-cb.

So, given this square matrix:

$$\begin{bmatrix} 2 & 4 \\ -3 & 6 \end{bmatrix}$$

The determinant is ad-cb = (2)(6) - (-3)(4) = 12 - (-12) = 24.

We won't *use* determinants until later lessons, so here's a hint: Matrices are very helpful for solving multiple simultaneous (linear) equations. These solutions are found by dividing various values by the determinant. If the determinant is zero, there is no (single) solution for the matrices, because you're not allowed to divide by zero.