Advanced Math: Notes on Lessons 41-44
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Lesson 41: Reciprocal Trig Functions / Permutation Notation
Basically, memorize the definitions of the reciprocal trig functions:
sec x = 1/cos x, csc x = 1/sin x, and cot x = 1/tan x.

The number of permutations (possible sequences) of a set of *n* distinct things, taken *r* at a time, is:

$$_{n}P_{r} = P(n,r) = \frac{n!}{(n-r)!}$$

This is just an extension of what we did earlier. If there are *n* distinct things, at the first position we have *n* choices, at the next one we have *(n-1)*, and so on, for *r* positions.

Note: I personally prefer the P(n,r) notation because (1) it's easier to write, (2) the $_{n}P_{r}$ notation can be confusing when it's combined with other values that have subscripts, and (3) a lot of software has trouble with the first notation, making it a pain to type. OpenOffice.org (the program I use to write these notes) has an "lsub" operator that directly supports this sort of thing ("lsub" means "subscript to the left of the expression") - but a lot of software does not have convenient operators to do this. You are welcome to use $_{n}P_{r}$ - Saxon will - but feel free to use any of the various accepted notations for permutation (there's quite a collection of alternatives!), including P(n,r).

Since there are 26 distinct uppercase letters, the number of distinct 3-letter words that can be made from those letters, when letters are not allowed to repeat, are:

$$_{26}P_{3} = P(26,3) = \frac{26!}{(26-3)!} = \frac{26!}{23!} = 26 \times 25 \times 24 = 15600$$

Lesson 42: Conic Sections / Circles / Constants in Exponential Functions
Conic sections: See the drawings in Saxon, I can't do better. Basically, if you start with a 3-dimensional circular code, you can cut it with a plane and produce various "images" where they intersect, including a circle, ellipse, parabola, or hyperbola.

The standard form for the equation of a circle is $x^2+y^2=r^2$. But that only works if the circle is centered at the origin. Otherwise, the standard form is $(x - x_{center})^2 + (y - y_{center})^2 = r^2$. Another common format for circle equations is the "general form"; just move the terms so that 0 is on the right-hand side.

If you're analyzing an exponential function where there's also a constant in the exponent, it's often better to move the constant "out" so that you have only the function parameter. E.G.,
$f(x) = 2^{2x} = (2^2)^x = 4^x$.

Lesson 43: Periodic Functions / Graphs of Sin and Cos

Make sure you can quickly re-draw the graphs of sin and cos. Once you can do one, you can do the other; they're just shifted from each other. sin 0 = 0, but cos 0 = 1. [Show drawings in class]

Lesson 44: Abstract Rate Problems

You've seen abstract rate problems with distances; this lesson just shows that you can use exactly the same approach with other rates, including price per unit or the time per job. A "rate" is just one value divided by another. Since there are two ways to divide numbers, figure out which number you want to use directly and place *that* one "on top". In many problems you'll have "old" and "new" situations; figure out all the "old" values, labeling, them, and then figure out the "new" values (labeling them too).

Here's a trivial example: Today you can buy a set of s soda cans for a total of c cents. If tomorrow the whole set of s sodas cost an x more cents, then at that rate, how many sodas can you buy for y cents?

Sodas = s
Price for set of s sodas = c
Original Rate in sodas/cent = s/c sodas/cent ← Notice we put # sodas on top

Let's figure out tomorrow's rate, which is expressed in terms of buying s number of sodas:
New price for set of s sodas = c + x
New Rate = (#sodas) / (# cents for that #sodas) = s/(c+x) sodas/cent
But we won't buy exactly s sodas. To find a total number of items purchased, you multiply the rate * price paid to get the total number of items purchased:

$$(Rate) \times (Price\ paid) = Number\ purchased$$

$$(Rate) \times (Price\ paid) = \frac{s}{c+x} \frac{sodas}{\cancel{cents}} \times y\, \cancel{cents} = \frac{sy}{c+x}\ sodas\ purchased$$

The key here is to include all the units, and *make sure they cancel* (like the cents do, above), and *make sure that the final unit is the one you want*. By checking the units you'll avoid many likely mistakes.

Extra time: Differentials

This is a surprisingly easy week. So, I thought I'd spend a few minutes introducing an idea not in the book: differentials. There won't be a test on this, so don't worry if you don't get the details. My intent it to basically hint at a basic idea that you'll really learn about later, so that when you *do* study them, it'll be easier to grasp (because you've heard a about them before).

Calculus is primarily about studying two things: differentials and integrals. Today, you'll learn what a differential is. Differentials aren't really complicated at their heart. They're just the slope of a line, at various points. So let's learn what they are by analogy...

Imagine you're on a weird rollercoaster, described by the function $f(x)=2x^2+3$. You should already be able to draw this. But , what is the slope of the rollercoaster at various points (i.e., the "slope of the tangent")?

The answer is, in fact, another function. After all, the slope of a function at any point is usually different from place to place. This "another function" is called the "differential" of the first function. A differential is just another function that tells you the slope of the first function at any point you'd like to know. Another name for "slope" is "rate of change", so you could also say that the differential tells you the *rate of change* of a function. It's the same thing.

Let's name the differential of our example as g(x). It turns out that in this example g(x)= 4x. (For the moment, don't ask how I determined that.) This means that at x=1, the slope is g(1)= 4(1) = 4. So at x=1, we're going up 4 units for every one unit to the right. That's a steep line; you can use arctan to find the angle, in this case arctan(4/1) = 62.5°.

But how would you determine the differential of a function? What we want is the slope of a line at a single point; we can find that slope by starting with an approximation and getting closer and closer to it [walkthrough in class]. Basically, to find a slope at some point x, you figure out the slope between x and some later position x+h, then make the two points closer and closer by making h smaller and smaller. You've already heard about "limits"; basically, find out what the value of the slope approaches as h approaches 0. [Walkthrough graph on board]

$$\textit{differential of } f = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

The weird thing is that it *looks* like we have a division-by-zero on the right. But in fact, we don't; we never *actually* divide by zero. We just try to figure out what the whole expression gets close to, as h gets *closer and closer* to zero, and then report *that*. This trick (for avoiding divide-by-zero) makes calculus work.

So here's an example of how to find a differential this way. If $f(x)=2x^2+3$, then

$$\textit{differential of } f = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(2(x+h)^2+3)-(2x^2+3)}{h}$$
$$= \lim_{h\to 0} \frac{(2(x^2+2xh+h^2)+3)-(2x^2+3)}{h} = \lim_{h\to 0} \frac{(2x^2+4xh+2h^2+3)-(2x^2+3)}{h}$$
$$= \lim_{h\to 0} \frac{4xh+2h^2}{h} = \lim_{h\to 0} 4x+2h = 4\text{x}$$

In the last steps, we can cancel out the h variables because h is really close but not equal to zero. In the last step, what we want to find out is, "as h goes to zero what does value get close to" - and as h goes zero, 2h goes to zero, so increasingly vanishes, and what we are left with is that the expression gets closer and closer to the rest (in this case, 4x).

I'm not going to ask you to actually *calculate* differentials for now. The key thing to understand right now is that a *differential* is simply *another* function, created from some first function, that tells you the *slope* or *rate of change* of the first function at any position. And *rates of change* are everywhere. For example, if a function describes the *position* of an object, then its differential (rate of change) is the *speed* of the object.

Calculating differentials from first principles can be hard work... presenting the risk of getting confused and missing the main point. *That* is what I'm trying to guard you against right now. So *don't* get confused. Calculus at its basis is actually pretty simple. One part is "differentials", which is simply just finding the slope of a line. Calculating them can be work, but if you know what you're trying to *accomplish* ahead-of-time, you should do better. Sometime I'll talk about the other part of Calculus (integrals).