Advanced Math: Notes on Lessons 106-109

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Lesson 106: Translations of Conic Sections / Equations of the Ellipse / Equations of the Hyperbola

If you want to translate (move without rotation) some equation to center on (h,k) instead of (0,0), just replace x with “x-h” and y with “y-k”. Since a circle of radius r centered at (0,0) has the equation:

$$x^2 + y^2 = r^2$$

The equation of a circle of radius r centered at some arbitrary point (h,k) is:

$$(x-h)^2 + (y-k)^2 = r^2$$

An ellipse centered at the origin is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

So translating that ellipse to center on (h,k) produces:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Remember, our convention is to label “a” as whatever the larger value is. If it’s under the x-term, it will have a horizontal major axis. But the horizontal and vertical major axis have the same form.

Similarly, a hyperbola translated so it’s centered on (h,k) produces:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Sometimes you’ll have an equation in this form instead:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

To find information such as where it’s centered, you’ll need to complete the square to transform it back to the (x-h,y-k) format (above). Once you get there, you’ll know what it is (ellipse, hyperbola, etc.), as well as where it’s centered. Here’s an example of doing this:

$9x^2-36x+16y^2+96y+36=0$	*Initial value*
$9(x^2-4x+\_\_)+16(y^2+6y+\_\_)+36=0$	*Placeholder for completing square*
$9(x^2-4x+4)-9(4)+16(y^2+6y+9)-16(9)+36=0$	*Completed square*
$9(x-2)^2-9(4)+16(y+3)^2-16(9)+36=0$	*Replaced with square*
$9(x-2)^2+16(y+3)^2=144$	*Simplified*
$\frac{(x-2)^2}{16}+\frac{(y+3)^2}{9}=1$	*Divide so '1' is on RHS*
$\frac{(x-2)^2}{4^2}+\frac{(y+3)^2}{3^2}=1$	*Ellipse, center (2,-3), horiz. major axis*

Lesson 107: Convergent Geometric Series (and hints at "limit")

As noted in lesson 104, you can compute a geometric series (a sum of the elements of a geometric sequence) this way:

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-2} + a_1 r^{n-1}$$

Multiply equation by r:

$$r S_n = \quad r a_1 + a_1 r^2 + a_1 r^3 + a_1 r^4 + \ldots + a_1 r^{n-1} + a_1 r^n$$

Find first - second:

$$S_n - r S_n = a_1 - a_1 r^n$$

$$(1-r) S_n = a_1 (1 - r^n)$$

$$S_n = \frac{a_1 (1 - r^n)}{1 - r}$$

But imagine that r^n is very close to zero; if that's so, then $1-r^n$ will be very close to 1. In fact, as r^n gets closer to zero; the expression $1-r^n$ will get closer to 1. This means that the *limit* of $1-r^n$ is 1 as r^n approaches 0.

This "limit" idea is a *critical* step in the road to calculus. Fundamentally, we can't divide by zero, and that's still true. For many years it appeared that calculus required division by zero. It turns out that it does not; calculus just needs this idea that values *approach* other values in certain conditions, aka the limit. So now is a good time to introduce this idea of limits.

In this case, if you have an *infinite* sequence, can you still add up all the values? The answer is yes, if $|r| < 1$ and $r \neq 0$. In this case, as n gets larger, r^n approaches 0. As n gets larger and larger, tending towards infinity, r^n gets smaller and smaller, tending towards 0. In the limit, as n approaches infinity, r^n approaches 0, resulting in this equation:

$$S_n = \frac{a_1}{1 - r} \quad where\ r \neq 0, |(r)| < 1$$

Lesson 108: Matrix Addition and Multiplication

The *order* of a matrix is simply number-of-rows *x* number of columns. So this is a 2x3 matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

You can "add two matrices" only if they have exactly the same order (same number of rows and columns), and the rule is trivial: Just add the corresponding elements. So:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 3 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1+7 & 2+8 & 3+9 \\ 4+3 & 5+8 & 6+2 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 7 & 13 & 8 \end{bmatrix}$$

Since adding any two numbers is commutative (a+b=b+a for normal numbers), matrix addition is commutative as well. So for two matrices A and B, A+B = B+A.

You can multiply a matrix by a single number; the result is called a "scalar product", and it's computed in a similar way - just multiply each matrix element by that number. For example,

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\times 3=\begin{bmatrix}1\times 3 & 2\times 3 & 3\times 3\\4\times 3 & 5\times 3 & 6\times 3\end{bmatrix}=\begin{bmatrix}3 & 6 & 9\\12 & 15 & 18\end{bmatrix}$$

Computing a scalar product is commutative, that is, for scalar k and matrix A, kA = Ak. We'll see why I'm harping on commutativity in a moment.

Multiplying matrices

Now we come to the time-consuming part: Multiplying two matrices together. You can *only* multiply two matrices if the first matrix's order ends with the same number that the second matrix's order begins with. Thus, a 4x3 matrix can be multiplied by a 3x7 matrix (because the 3's match), but a 3x4 matrix cannot be multiplied by a 3x7 matrix (the "4" of "3x4" does not match the "3" in "3x7").

The resulting matrix size is the "outermost" numbers of the order, that is, the first matrix order's first number *x* second matrix order's last number. So a 4x3 matrix multiplied by a 3x7 matrix will produce a 4x7 matrix.

Now we come to the part that's hard to describe - how to calculate the value of each element that results from matrix multiplication. Saxon suggests that you lay out the matrix as values of $SP_{x,y}$, where "x" is the current row and "y" is the current column. To compute the values, multiply the x-row elements of the left-hand matrix with the corresponding y-column elements of the right-hand matrix, and add them up.

$$\begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}\times\begin{bmatrix}g & h\\i & j\end{bmatrix}=\begin{bmatrix}SP_{11} & SP_{12}\\SP_{21} & SP_{22}\\SP_{31} & SP_{32}\end{bmatrix}=\begin{bmatrix}ag+bi & ah+bj\\cg+di & ch+dj\\eg+fi & eh+fj\end{bmatrix}$$

A key thing to note is that *matrix multiplication is **not**, in general, commutative.* That is, given two matrices A and B, AB is not necessarily equal to BA. Order matters!

Here's another example, this time with numbers:

$$\begin{bmatrix}2 & 3 & 4\\5 & 6 & 7\\0 & 2 & 0\end{bmatrix}\times\begin{bmatrix}1 & 5 & 6\\9 & 1 & 0\\8 & 0 & 0\end{bmatrix}=\begin{bmatrix}SP_{11} & SP_{12} & SP_{13}\\SP_{21} & SP_{22} & SP_{23}\\SP_{31} & SP_{32} & SP_{33}\end{bmatrix}=$$

$$\begin{bmatrix}2(1)+3(9)+4(8) & 2(5)+3(1)+4(0) & 2(6)+3(0)+4(0)\\5(1)+6(9)+7(8) & 5(5)+6(1)+7(0) & 5(6)+6(0)+7(0)\\0(1)+2(9)+0(8) & 0(5)+2(1)+0(0) & 0(6)+2(0)+0(0)\end{bmatrix}=\begin{bmatrix}61 & 13 & 12\\115 & 31 & 30\\18 & 2 & 0\end{bmatrix}$$

Typical mathematical notation shows matrix multiplication with the same notation as any other multiplication, but many programs use a special notation for matrix multiplication. E.g., wxMaxima uses "." for matrix multiplication but "*" for ordinary multiplication. On some other applications/calculators you must use special functions to do matrix multiplication, e.g., OpenOffice.org's Calc spreadsheet application uses MMULT(matrix1;matrix2) for matrix multiplication.

Matrix Identities

With ordinary (scalar) numbers, x+0=x. Similarly, if you add a matrix to the "additive identity" matrix, it won't change. The "additive identity" matrix is just a matrix with 0 as every element, and has the same order as the matrix it's being added to (since orders must match for addition).

With ordinary scalar numbers, $x \cdot 1 = 1 \cdot x = x$. Similarly, a matrix with a diagonal of 1 from top left to bottom right, and 0 everywhere else, acts like "1" in an ordinary scalar multiplication. This special form of a matrix, because it acts like "1" in ordinary multiplication (when you use the right order), is called the "multiplicative identity" or "identity" matrix. You can multiply any matrix by the identity matrix of the correct order, and produce the same original matrix. This is one of the few cases where order doesn't matter; the identity matrix can be first or second (i.e., it *does* commute). Here's a 3x3 identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

You should be able to work out the multiplication, for example, to show to yourself that:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Lesson 109: Rational Numbers

A "rational number" is just a number that can be written as a fraction (an integer divided by another integer). All of the integers Z are rational numbers, because you can write them as Z/1.

If you have a repeating decimal, then you have a rational number. You can calculate that number using the formula for calculating infinite geometric series in lesson 107.